Increasing of entanglement entropy from pure to random quantum critical chains.

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It is known that the entropy of a block of spins of size L embedded in an infinite pure critical spin chain diverges as the logarithm of L with a prefactor fixed by the central charge of the corresponding conformal field theory. For a class of strongly random spin chains, it has been shown that the correspondent block entropy still remains universal and diverges logarithmically with an "effective" central charge. By computing the entanglement entropy for a family of models which includes the N-states random Potts chain and the Z_N clock model, we give some definitive answer to some recent conjectures about the behaviour of this effective central charge. In particular, we show that the ratio between the entanglement entropy in the pure and in the disordered system is model dependent and we provide a series of critical models where the entanglement entropy grows from the pure to the random case.

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The effects of quenched randomness on quantum critical systems have been the subject of an intense experimental and theoretical activity for many years [1]. Disordered one dimensional (1D) quantum models provide an important framework to study these effects because the quantum fluctuations are enhanced in this dimension and the corresponding pure systems can be studied using a wide variety of approaches, such as Bethe-ansatz solutions and conformal field theories techniques. For systems displaying strong spatial inhomogeneities, many results can be obtained by the application of the strong disorder renormalization group (SDRG) method [2] introduced by Ma, Dasgupta and Hu [3]. For a wide variety of random quantum critical models, ranging from the random quantum chains with discrete symmetries to the random Heisenberg models on fractal lattices [4, 5, 6, 7, 8, 9, 10, 11, 12], it has been shown that the effective disorder grows indefinitely at larger scales. This flow to strong disorder occurs in particular for the random spin 1/2-Heisenberg chain, the random XX chain and the random transverse field Ising model (RTFIM). The physics of these systems is captured by the so called infinite random fixed point (IRFP). Some of the main features of this point are a strong dynamical anisotropy and an extreme broad distribution of physical quantities which manifests through drastically different behaviour between average and typical correlation functions.

Recently, Refael et Moore [13] have used the particular properties at the IRFP of these models to compute the entanglement of a block of spins of the corresponding chain. The entanglement of a segment of L spins with the remainder is defined as the von Neumann entropy of the reduced density matrix $\hat{\rho}_L$ [14]:

$$\mathcal{S}(L) = -Tr\hat{\rho}_L \ln \hat{\rho}_L. \qquad (1)$$

In an infinite quantum critical chain without disorder, the entanglement of the segment with the remaining sites is proportional to $\ln L$ with a coefficient depending on the central charge c of the associated conformal field theory [15, 16, 17]:

$$\mathcal{S}(L) \sim \frac{c}{3} \ln L \qquad (2)$$

The main result of [13] is that the universal logarithmic scaling of the entanglement entropy still holds at the IRFP of these models, where the conformal invariance is lost. In particular they showed that the entanglement entropy $\mathcal{S}_{RTFIM}$, $\mathcal{S}_{RH}$ and $\mathcal{S}_{RXX}$ respectively for the RTFIM, the random Heisenberg and XX chain is:

$$\begin{aligned} \mathcal{S}_{RH}(L) = \mathcal{S}_{RXX}(L) \ &\sim\ \frac{\log 2}{3} \ln L \\ \mathcal{S}_{RTIFM}(L) \ &\sim\ \frac{\log 2}{6} \ln L \end{aligned} \qquad (3)$$

Numerical evidences for the logarithmic scaling have been obtained for the random Heisenberg and XX chain in [18, 19].

Analogously to the role played by the central charge, the prefactor of the logarithmic divergence can be a good estimator to characterise the universality class of the random chains. The continuum limit of the pure Heisenberg (H) and XX chains (XX) are described by the $c = 1$ free boson theory (with different compactification radius) while the pure quantum Ising (TFIM) is described by the $c = 1/2$ free fermion theory. From (2), one has $S_H(L) = S_{XX}(L) \sim \ln L/3$ and $S_{TFIM}(L) \sim \ln L/6$. Hence, the ratio between the random and pure logarithmic prefactor is, for these two models, the same. This has suggested the intriguing possibility that the coefficient appearing in the entropy scaling of any random fixed point is the product of the central charge of the associated pure critical system and an universal number ($\ln 2$). The findings (3), from which the entanglement of the pure systems turns out to be reduced by disorder, suggest also a loss of entanglement along a general SDRG flow. Note that the loss of entanglement entropy was proposed in [20] as a signal of the irreversibility of the renormalization group (RG) flow . This conjecture has been explored for the quantum Ising chain by varying the transverse magnetic field and it was also found to

be consistent with the predictions for a Luttinger liquid with one single impurity [21].

In this Letter we adopt the analysis of Refael and Moore to study the entanglement entropy of a family of random quantum critical chains, which includes the N-state random quantum Potts chain and the Z_N clock model. By means of the SDRG procedure, we show that all these random chains are attracted by the infinite random fixed point of the RTFIM. Computing the associated entanglement entropy, we show that that the correction from a pure conformal fixed point to the random fixed point depends on N and thus is model-dependent. Moreover we find that for the random parafermionic Z_N chains (defined later) with $N > 41$, the entanglement is greater than the one in the pure system, thus providing a counter-example to the above conjecture.

We consider a chain of spins with N states $|s_i\rangle = |0_i\rangle, |1_i\rangle, \cdots, |(N-1)_i\rangle$ where i indicates the lattice site. The random quantum spin model we study is defined by the Hamiltonian $\mathcal{H}_N$:

$$\mathcal{H}_N = \quad -\sum_i J_{i,i+1} \sum_{n=1}^{N-1} \alpha_n \left(\bar{S}_i^z S_{i+1}^z\right)^n - \\ -\sum_i h_i \sum_{n=1}^{N-1} \alpha_n \Gamma_i^n. \qquad (4)$$

in terms of the operators $S^z = e^{2i\pi/Nq}\delta_{q,q'}$ (with $q, q' = 0 \cdots N-1$), their hermitian conjugates $\bar{S}^z$ and spin raising operators $\Gamma|s\rangle = |s+1, \bmod N\rangle$. The couplings $J_{i,i+1}$ and the transverse fields h_i are independent positive random variables drawn from some distribution. The coefficients α_n satisfy $\alpha_n = \alpha_{N-n}$ to assure the Hamiltonian to be hermitian and they are assumed to be disorder independent. For each set of values $\{\alpha_n\}$, it exists a transformation from site variables to bond variables $\bar{S}_i^z S_{i+1}^z = \Gamma_i^*$; $\prod_{j\leq i} \Gamma_j = S_i^{z,*}$ which yields the same Hamiltonian with the J's and h's interchanged. The model (4) is thus provided of a duality transformation. In the following we will always assume the couplings and fields to have initial equal distribution.

If all the coefficients α_n are equal to unity, $\alpha_n = 1$ for $n = 1 \cdots N-1$, the Hamiltonian (4) has a permutational symmetry S_N and it defines the random quantum N-states Potts model. The case $\alpha_n = \delta_{1,n}$ corresponds to the random quantum Z_N clock models. Senthil and Majumdar [11] showed that these two models, with ferromagnetic couplings, are attracted to the IRFP of the RTFIM. As they noticed, this implies that the scaling behaviour of many physical quantities (such as the magnetisation or the mean correlation functions) depend on statistical properties entirely given by the IRFP distributions which are N-independent. So, despite the fact that in the absence of disorder these systems have different type of correlations, the low-energy behaviour of the random systems presents N-independent scaling functions and exponents. On the other hand, the entanglement entropy at the IRFP of these models is, as we show later, sensitive to the number of spin states. This quantity can then discriminate between models with different N.

Here we focus also on another class of random quantum spin chains defined by the Hamiltonian (4) with $\alpha_n = \sin(\pi/N)/\sin(\pi n/N)$. The interest in this model is motivated by the following facts: i) the corresponding pure model is critical at the self dual line $J = h$ [23] and its fluctuations are governed by the Z_N parafermionic field theories with central charge $c_N = 2(N-1)/(N+2)$ introduced in [22] (the case $N = 2$ and $N = 3$ correspond to the Ising and the $N = 3$ state Potts model respectively). ii) The SDRG procedure can be performed for all this family of theories. iii) The scaling of entanglement entropy for the Z_N spin chain can be computed using the properties of the IRFP and compared with its value at the parafermionic conformal critical points. Henceforth we refer to this model as the Z_N parafermionic spin chain.

We apply the SDRG approach to the model (4) by assuming the coefficients $\alpha_n \leq 1$. The renormalization equations are obtained by successively eliminating the maximum of the amplitudes of the bonds $J_{i,i+1}$ and fields h_i, getting an effective Hamiltonian for low energy degrees of freedom. At each decimation step, if this maximum turns out to be the field h_i, the corresponding spin $|s_i\rangle$ is frozen in the state $|s_i\rangle = 1/\sqrt{N}(|0_i\rangle + |1_i\rangle + \cdots + |(N-1)_i\rangle)$ which is the ground state of the dominant term $-h_i \sum_{n=1}^{N-1} \alpha_n \Gamma_i^n$. This generates an effective new coupling $\tilde{J}_{i-1,i+1}$ between adjacent sites. The coefficients α_n enter into the renormalization as well. Degenerate second order perturbation theory yields:

$$\tilde{J}_{i-1,i+1} = \frac{J_{i-1}J_{i+1}}{\kappa_1 h_i} \qquad \tilde{\alpha}_n = \alpha_n^2 \frac{\kappa_1}{\kappa_n}. \qquad (5)$$

where the $\tilde{\alpha}_n$ are the renormalized coefficients and

$$\kappa_n = \frac{1}{2} \sum_{m=1}^{N-1} \alpha_m \left[1 - \cos\left(\frac{2\pi mn}{N}\right)\right]. \qquad (6)$$

On the other hand, if the maximum is the coupling $J_{i,i+1}$, we set spin $|s_i\rangle$ and spin $|s_{i+1}\rangle$ to have the same value, $|s_i\rangle = |s_{i+1}\rangle$. The effective field $\tilde{h}_i$ acting on the ferromagnetic cluster $|s_i\rangle \otimes |s_i\rangle$ is:

$$\tilde{h}_i = \frac{h_i h_{i+1}}{\kappa_1 J_{i,i+1}}, \qquad (7)$$

as it could be guessed by duality.

Due to our initial assumptions on the coefficients $\{\alpha_n\}$, we have $\kappa_1 \geq 1$ for $N \geq 2$. This assures that, during the renormalization procedure, the energy scale is decreasing: the fixed point of the problem is thus expected to be strongly attractive, even for weak disorder. It is important also to remark that the disorder drives the system away from the parafermionic critical point but the self-duality is preserved during the decimation procedure.

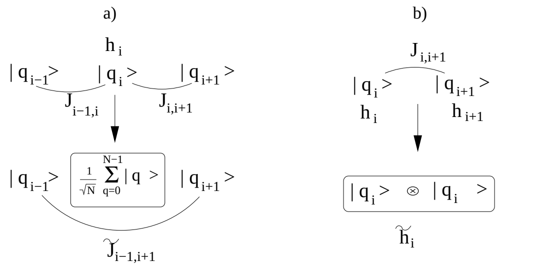

FIG. 1: Schematic picture of the two kind of decimation step: a) Field decimation. b) Coupling decimation.

Starting with the Potts values $\alpha_n = 1$ for $n = \cdots N-1$, the coefficients do not renormalize during the decimation procedure. In other words the permutational symmetry S_N is not broken along the flow of the couplings. On the contrary, by applying iteratively Eq.5 and Eq. 6, the coefficients $\{\tilde{\alpha}_n\}$ defining the Z_N parafermionic spin chain renormalize quickly to the values $\tilde{\alpha}_n \to \delta_{1,n}$. We can thus conclude that the random parafermionic Z_N spin chain and the random Z_N clock model share the same low-energy behaviour. In particular the couplings and fields distribution is attracted towards the IRFP distribution.

The analysis of Refael and Moore can then be directly generalised to the more general random quantum Hamiltonian (4). The critical fixed point distribution yields a concentration n_Γ of cluster at the log-energy scale Γ which scales as $n_\Gamma \sim 1/\Gamma^2$. Hence the ground states of all these models present clusters which form over an average length $\lambda \sim \Gamma^2$. Let us consider in detail a cluster of s spins. We want to compute the entanglement entropy of a subsystem of r spins with the remaining $s-r$ spins of the cluster. If the cluster has been decimated by the field, it is frozen in the state $|\psi\rangle$:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \overbrace{|q\rangle \otimes |q\rangle \otimes \cdots \otimes |q\rangle}^{\text{s times}}. \tag{8}$$

The entries of the reduced density matrix $\hat{\rho}_r$ can be defined in the spin basis ϕ_r^i, $i,j = 1..2^r$ as:

$$\langle \phi_r^i | \rho_r | \phi_r^j \rangle = \sum_{k,l=1}^{2^{s-r}} \langle \phi_r^i | \otimes \langle \phi_{s-r}^k | \psi \rangle \langle \psi | \phi_{s-r}^l \rangle \otimes | \phi_r^j \rangle, \tag{9}$$

where one traces over the basis ϕ_{s-r}^i of the other $s-r$ spins of the cluster. Using the wave function (8) in (9), one gets that the entanglement of a subsystem of r spins with the remaining $s-r$ ones of a frozen cluster is $\log N$ for each r and s. Returning to the entanglement of a segment L embedded in an infinite chain, the frozen clusters which are totally inside or outside the segment L do not affect the total entanglement. The only contribution comes from the number of decimated clusters which cross the two edges, each of such clusters adding $\log N$ to the total value of the entanglement (see Fig.2). One has then

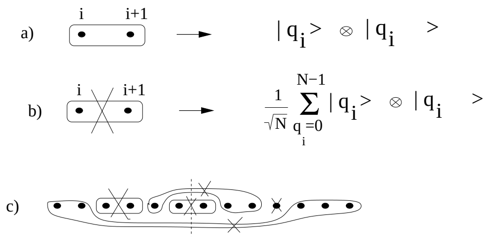

FIG. 2: a) The state describing two spins decimated by the couplings. b) The state of a cluster of two spins after the field decimation. c) The contribution of the entanglement is given by the field decimated cluster which cross an edge of the segment (dashed line). In this example, we have three of such clusters.

$$\mathcal{S}_N^{dis}(L) \sim 2 \, \log N \, p \, \langle D \rangle_L \tag{10}$$

where $\mathcal{S}_N^{dis}(L)$ is the entanglement of a segment L in the disordered quantum chain (4), $\langle D \rangle_L$ is the average number of decimation which occur over an edge of the segment L, p is the probability to have a field decimation instead of a coupling one. Due to the self-duality of these models, preserved along the SDRG, one has $p = 1/2$. The value of $\langle D \rangle_L$ does not depend on N: it is entirely given from the fixed point distribution and it has been computed in [13] where it was found $\langle D \rangle_L = 1/3 \log L$. Collecting all these results we finally arrive at the value of entanglement for the family of random models (4) with $\alpha_n \leq 1$:

$$\mathcal{S}_N^{dis}(L) \sim \frac{\log N}{6} \log L. \tag{11}$$

The entanglement $\mathcal{S}_N^{dis}(L)$ depends thus on the number of states N. Hence, differently from other quantities which are N-independent, it represents a good estimator to characterise the universality class of the random chains. Note that this quantity alone is not sufficient to identify a random phase transitions as it can be seen, for instance, from the fact that the random quantum Potts model with symmetry S_N have the same entanglement of the models with a lower symmetry Z_N. This is analogous to what happens in conformal field theory where different models can have the same central charge.

Taking into account the value of the central charge c_N, the entanglement $\mathcal{S}_N^{pure}(L)$ of the pure Z_N parafermionic spin chain is

$$\mathcal{S}_N^{pure}(L) \sim \frac{2(N-1)}{3(N+2)} \ln L. \tag{12}$$

Comparing the values $\mathcal{S}_N^{pure}$ and $\mathcal{S}_N^{dis}(L)$, we can rule out the existence of a model-independent correction which links the values of the entanglement of the pure and of

the disordered system. Actually, there is no any clear relationship between the value of the entanglement prefactor in the disordered chain and the central charge associated to the pure chain. It would be interesting to check other observables, for instance the scaling of the free energy at the IRFP, which could clarify better if the value of this prefactor behaves as an "effective central charge", as recently proposed [13, 18].

It is interesting to note that :

$$\mathcal{S}_N^{dis}(L) > \mathcal{S}_N^{pure}(L) \quad \text{for} N > 41, \tag{13}$$

thus invalidating the recent conjecture of a loss of entropy from pure to disordered critical chain. This conjecture followed along the lines of the c-theorem [24] which states the existence of a function of the couplings monotonically decreasing along the renormalization group. For this reason it is interesting to mention the fact that, in the presence of disorder, the unitarity (one of the key hypotheses of the c-theorem) is in general lost. This is known to happen, for instance, in the case of the two-dimensional random bond 3-states Potts model which flows towards a random fixed point where the conformal symmetry is restored [25, 26]. In this case, one can show that the energy-energy operator, which determines the properties of the RG equations, gets a negative norm in the disorder limit. This explains the fact that, for this model, the central charge associated to the disordered random fixed point is greater than the one associated to the pure fixed point. On the other hand, the loss of unitarity does not necessarily imply the increasing of the central charge, as it has been explicitly shown for instance in [27], where a particular class of disordered conformal field theories was studied. In an analogous way, we think that, in the presence of disorder, one cannot predict the behaviour of the entanglement on the basis of a generalised $c-$theorem, as our results clearly show.

To conclude, in this Letter we have shown that the entanglement entropy of a class of random quantum critical chains, including the random quantum Potts and the random quantum clock chain, obey an universal logarithmic scaling. We have confirmed thus the entanglement as an important quantity to characterise (although not completely) the universality class of this entire series of disordered critical models. Moreover our results provide a definitive answer to some recent hypotheses on the entanglement entropy in the random quantum critical chains. By considering the Z_N parafermionic spin chains, we could compare the value of the central charge of the pure system with the value of the entanglement entropy of the disordered one for each value of N. We found that the ratio between the central charge and the prefactor of the entanglement logarithmic scaling of the disordered models depends on N. We could thus exclude the existence of a model-independent correction between these two quantities. Moreover, we provide some explicit models where the entanglement of the pure system is smaller than the entanglement of the disordered one. Thus the recent conjecture about a generalised c-theorem concerning the entanglement does not hold when the disorder is present.

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